Partiële differentiaalvergelijkingen, WIPDV-07 2010/11 semester II b

Re-examination, August 23rd, 2011.

Name	**Student number**

Notes:

- You may use one sheet (single side written) with notes from the lectures.
- During the exam it is NOT permitted to consult books, handouts, other notes.
- Numerical/graphic calculators are permitted, symbolic calculators are NOT permited.
- Devices with wireless internet connection and/or document readers are NOT permitted.
- To pass the exam, You need to gather at least half of the total points at the final exam.
- Hint: please describe the solution procedures in full details, not only the results.

TEST (to be returned by 12:00)

1. (a) **[pts 5]** Show that the system

$$u_x = 3x^2y + y,$$
$$u_y = x^3 + x,$$

with the initial condition $u(0,0) = 0$ has a unique solution.

(b) **[pts 5]** Prove that the slightly different system

$$u_x = 2.999999x^2y + y,$$
$$u_y = x^3 + x,$$

has no solution at all.

2. Consider the equation

$$u_{xx} + 2u_{xy} + [1 - q(y)]u_{yy} = 0,$$

where

$$q(y) = \begin{cases} -1, & y < -1, \\ 0, & |y| \leqslant 1, \\ 1, & y > 1. \end{cases}$$

(a) **[pts 3]** Compute the domains where the equation is hyperbolic, parabolic, and elliptic.

(b) **[pts 2]** Draw graphically the domains where the equation is hyperbolic, parabolic, and elliptic.

3. Let $u(x,t)$ be the solution of the wave problem defined on the whole real line

$$u_{tt} - 9u_{xx} = 0, \quad -\infty < x < \infty, \ t > 0,$$
$$u(x,0) = f(x) = \begin{cases} 1, & |x| \leqslant 2, \\ 0, & |x| > 2, \end{cases}$$
$$u_t(x,0) = g(x) = \begin{cases} 1, & |x| \leqslant 2, \\ 0, & |x| > 2. \end{cases}$$

(a) **[pts 4]** Find the value of the solution u in the point $(0, \frac{1}{6})$, that is: find $u(0, \frac{1}{6})$.

(b) **[pts 5]** Discuss the behavior of the solution at large time, that is: find $\lim_{t\to\infty} u(x,t)$.

(c) **[pts 6]** Find the maximal value of $u(x,t)$, and the points where this maximum is achieved.

4. **[pts 8]** Using the method of separation of variables, derive step by step the complete solution (including the expression of the coefficients) for the problem of a vibrating string with fixed ends:

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = 0, && 0 < x < L, \quad 0 < t, \\ &u(0,t) = u(L,t) = 0, && t \geqslant 0, \\ &u(x,0) = f(x), && 0 \leqslant x \leqslant L, \\ &u_t(x,0) = g(x), && 0 \leqslant x \leqslant L. \end{aligned}$$

5. (a) **[pts 2]** Recall the definition of harmonic function.

(b) **[pts 3]** Compute the expression of the function

$$u(x,y) = \frac{x}{x^2 + y^2}.$$

in polar coordinates (r, θ).

(c) **[pts 4]** Show that u defined at point b) is harmonic. You may use the expression u either in cartesian coordinates or in polar coordinates, as you prefer.

6. **[pts 7]** Write the Fourier sine series of the function

$$f(x) = \begin{cases} 4 - 5x, & 0 < x < \frac{1}{2} \\ 5 - 5x, & \frac{1}{2} \leqslant x < 1 \end{cases}$$

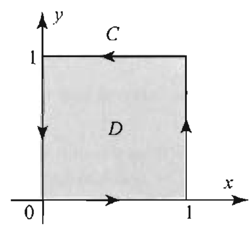
over the interval $[0, 1]$, and find the expression of the coefficients.

7. (a) **[pts 3]** Recall the first and the second Green's identities.

   (b) **[pts 4]** Using first Green's identity, calculate

$$\int_C y \frac{\partial x}{\partial n} ds$$

   where C is the region drawn below



   and the symbol $\frac{\partial}{\partial n}$ denotes the standard normal derivative.